Basics of MHD

{N.B.: Read Kulsrud, Chapt. 3, 4}

→ MHD Equations → Eulerian Fluid {1 Fluid, Large scale, slow}

① $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$ (Continuity)

→ Lorentz, $\underline{E}$?

② $\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \frac{\underline{J} \times \underline{B}}{c} + \underline{f}_{body}$

(momentum balance)

[frequently $\underline{f}_{body} = \rho \underline{g}$]

③ $\frac{d}{dt} S = \frac{\partial S}{\partial t} + \underline{v} \cdot \nabla S = 0$

{eqn. of state more general (isentropic fluid)

$S = C_v \ln (P/\rho^{\gamma})$
↑
entropy

[frequent form of equation of state]

④ $\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \left(\eta \underline{J}, \frac{\underline{J}}{\sigma} \right)$ (Ohms Law)

[resistivity η is usually most significant dissipation]

Ideal MHD

$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$

and ⑤ $\underline{\nabla} \cdot \underline{B} = 0$

⑥ $\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$

⑦ $\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J}$

} from Maxwell's EQNS. neglecting displacement current

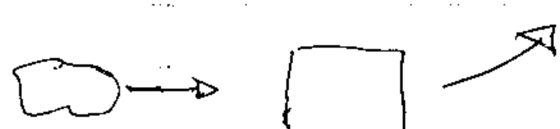
→ Meaning, Restriction, Validity

- MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

- Variants : Reduced MHD → strong $\underline{B_0}$ (tokamaks)
  - 2D MHD
  - E MHD → stationary ions (ICF)
  - FLR MHD, Reduced Braginsky } → MHD + additional effects (MFE, space)
  - hybrid → { bulk - MHD; hot species - kinetic (i.e. α's, energetics) }
  - ⋮

- MHD - Eulerian

"fluid element" ⟺ "glue"



fluid element　　fixed observer

here "glue" → collisions

applies $L > \lambda_{mfp}$

→ MHO is:
- 1 fluid - electrons and ions
- strongly collisional
- low frequency
- large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i};\ \omega_{pe,i};\ \nu_{ee,ei};\ \nu_{ii};\ \omega_{te,i}\ \text{etc.}$$

scales relevant:

$$L \gg \lambda_{De,i},\ \rho_{e,i},\ c/\omega_{pe,i},\ l_{mfp_{e,i}}$$

$$l_{mfp} < L$$

and

collisions isotropise, equilibrate $\underline{\underline{P}}$.

$$\left(\text{i.e.}\quad \underline{\underline{P}} \sim \int d^3v\ \tilde{v}_i \tilde{v}_j\ f(\underline{x}, \underline{v}, t) \right)$$

→ Some Specific Points:

- re: continuity ①;

$$\rho = m_i n_i + m_e n_e$$

{ total density / ion dominated }

i.e. (ions control fluid inertia)

- re: momentum balance ②;

→ $\underline{V} = \left(\int d^3v_i\, m_i \underline{v}_i f_i + \int d^3v_e\, m_e \underline{v}_e f_e\right) / \rho$

i.e. (ions control flow → $\rho \frac{d\underline{v}}{dt}$)

→ where has $\underline{E}$ gone? → $L \gg \lambda_D$ → quasi-neutrality

$$\rho_i \frac{d\underline{v}_i}{dt} = n_i q_i \underline{E} + n_i q_i \frac{\underline{v}_i \times \underline{B}}{c} + \cdots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = - n_e q_i \underline{E} - n_e q_i \frac{\underline{v}_e \times \underline{B}}{c} + \cdots$$

if add: → 0 cancel! ($n_i = n_e$ (quasi-neutrality)) → $\frac{\underline{J} \times \underline{B}}{c}$ (Lorentz force term in momentum balance)

note also: $\rho_i, \rho_e \rightarrow \rho$

→ re-writing the $\underline{J} \times \underline{B}$ force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

So can write:

$$\rho \frac{d\underline{v}}{dt} = -\underline{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

magnetic pressure (field energy density)

magnetic tension

a) What / Why "Magnetic Tension"?

$\underline{B} = B\,\hat{b}$ $\qquad B = |\underline{B}|$, $\hat{b} = \underline{B}/B$

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\,\hat{b}\cdot\underline{\nabla}(B\hat{b})$$

$$= B^2\,\hat{b}\cdot\underline{\nabla}\hat{b} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{2}\right)$$

(1) (2)

$\hat{b}\cdot\underline{\nabla}\,\hat{b} \to$ curvature of $\hat{b}$
(i.e. rate of change of $\hat{b}$ along itself)

$\equiv d\hat{b}/ds$

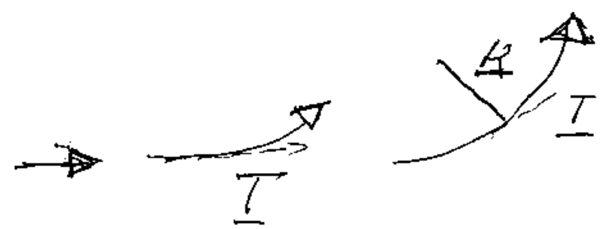
n.b. in general: curve: $\underline{x}(t)$

tangent: $\underline{T} = d\underline{x}/ds$

$(ds^2 = d\underline{x}\cdot d\underline{x})$ $\qquad s \equiv$ distance along curve curvature $\underline{K} = \dfrac{d\underline{T}}{ds} = \dfrac{d\underline{T}/dt}{ds/dt} = \dfrac{\dot{\underline{T}}}{|\underline{V}|}$

Now: $\underline{K} = \hat{\underline{b}} \cdot \underline{\nabla}\, \hat{b}$ → points in direction of turning of $\hat{b}$, orthogonal to $\hat{b}$

i.e.



$\underline{K} = d\,\underline{T}/ds$

∴ $\underline{K} = + \dfrac{\hat{\underline{n}}}{R_c}$ $\qquad R_c \equiv$ radius of curvature

as curved field line suggests "tension" → "magnetic tension".

b) What about ②?

But $\underline{J} \times \underline{B} \perp \underline{B}$ yet $\underline{\nabla}\left(\dfrac{B^2}{8\pi}\right)$ can have component along $\underline{B}$??

→ recombining total $\underline{J} \times \underline{B}$ gives:

$$-\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + B^2\frac{\vec{b}\cdot\underline{\nabla}\vec{b}}{4\pi} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$$

$$= -\nabla_\perp\left(\frac{B^2}{8\pi}\right) \cancel{-\hat{b}\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right)} \cancel{+\hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)} + B^2\frac{\vec{b}\cdot\nabla\vec{b}}{4\pi}$$

so
$$\boxed{\frac{\underline{J}\times\underline{B}}{c} = -\underline{\nabla_\perp}\left(\frac{B^2}{8\pi}\right) + B^2\frac{\vec{b}\cdot\underline{\nabla}\,\vec{b}}{4\pi}}$$

(3) $dE = đQ - p\,dV$ (Thermo)

$C_v\,dT = T\,dS - p\,dV$

$$\begin{cases} đQ = T\,dS \\ dE = C_v\,dT \end{cases}$$
(normalized)

$V = 1/\rho \qquad dV = -d\rho/\rho^2$

$$C_v\frac{dT}{T} = dS + \frac{d\rho}{\rho}$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln\rho^{1/C_v}$$

$$\therefore \boxed{S' = C_v\ln\left(T/\rho^{1/C_v}\right)}$$

$p = \rho T$

$$\Rightarrow \quad S = c_v \ln \left(P / \rho^{(c_v+1)/c_v} \right)$$
$$= c_v \ln \left(P/\rho^\gamma \right)$$

$\gamma = 5/3$, ideal gas

$\left(c_v = 3/2 \text{ (normalized)} \right)$

$$\frac{dS}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(P/\rho^\gamma \right) = 0$$

i.e.
$$\boxed{\frac{\partial}{\partial t}\left(P/\rho^\gamma \right) + \underline{v} \cdot \underline{\nabla} \left(P/\rho^\gamma \right) = 0}$$

eqn. of state

perfect $\left\{ \begin{array}{l} \text{homogeneity} \\ \text{stationarity} \end{array} \right.$ $\left(P/\rho^\gamma = \text{const.} \right)$

— "adiabatic equation of state"

(4) Ohm's Law — most sensitive part of MHD (since controlled by electrons)

MHD variants differ primarily in Ohm's Law
- Hall MHD → Hall term
- EMHD → electron inertia
- Bragiinsky / drift MHD → ∇P terms

etc., etc.

Ohm's Law $\Rightarrow$ subtract moments on electron equations
$\rightarrow$ electrons
$\left(\underline{J} = n\,e(\underline{V}_i - \underline{V}_e) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \frac{\eta}{S}\,\underline{J}$$

$\sim \nu_{e,i}$ $\rightarrow$ momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0 \quad \rightarrow \text{ field "frozen into" fluid}$$

5, 6, 7: Only 1 approximation:

$$\nabla\times\underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

$|\partial \underline{E}/\partial t| << |\underline{J}|$ $\rightarrow$ condition on ω ?

$$\rightarrow \quad \omega \frac{vB}{c} \overset{?}{<<} \frac{k}{c^{-1}} B$$

$\Rightarrow |\underline{v}|(\omega/k)/c^2 << 1$ is condition on ω.

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible?

$$\rho \frac{d\underline{v}}{dt} = n q \underline{E} + \cdots\cdots$$

and $q \neq 0$, as

$$n q = \frac{\nabla \cdot E}{4\pi}$$

$$\underline{E} = -\frac{\underline{v} \times \underline{B}}{c}$$

so

$$n q \underline{E} = +\left(\frac{\underline{v} \times \underline{B}}{c}\right) \underline{\nabla} \cdot \left(\frac{\underline{v} \times \underline{B}}{c}\right) \neq 0 \quad !$$

but

$$\sim \frac{v^2}{c^2} B^2 k$$

$$\sim \frac{v^2}{c^2} (\underline{J} \times \underline{B}) \longrightarrow \text{negligible, if } v^2/c^2 \ll 1.$$

Thus, yes indeed it does!

→ Putting it together:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J} \quad , \quad \underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

⇒ the induction equation for $\underline{B}$ evolution ...

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) – $\underline{V}(\underline{X}, t)$, $\underline{B}(\underline{X}, t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

so
$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

This brings us to ......

→ What Does "MHD", as a system, really mean ....?

Supplement

→ Deriving MHD

→ MHD is derived from 2-fluid equations

- first discuss 2 fluid derivation from Boltzmann
- then, discuss reduction to one-fluid MHD (i.e. approximations/limitations — especially in Ohm's Law)

→ deriving fluid equations

Have in general, Boltzmann eqn

$$\underbrace{\frac{\partial f}{\partial t}}_{①} + \underbrace{\underline{v}\cdot\underline{\nabla} f}_{②} + \underbrace{\frac{q}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\underline{\nabla} f}_{③} = \underbrace{C(f)}_{④}$$

④ collision operator

and can assign time scale

① ↔ ω → frequency

② ↔ $V_{Th}/L_{\parallel}$

↳ relevant parallel scale

③ $\frac{q}{m}\frac{E}{\Delta v}$ $\Delta v \sim v_{Th} \rightarrow$ non-resonant
$\Delta v \sim \Delta v_{Th} \rightarrow$ resonant

NL scattering rate (→ small, usually)

④ ν_{eff} – collision frequency

For "fluid description", need:

→ $\nu_{eff} > v_{Th}/L_{\parallel}$

i.e. short mean free path limit

or

→ $\omega > v_{Th}/L_{\parallel}$ → a/a' gyrokinetic KSAW, where $\nu \rightarrow 0$

i.e. "fluid" ↔ blob / fluid element of particles

→ what holds blob together?
(i.e. prevents dispersal?)
⇒ collisions (i.e. particles collide and scatter prior dispersal)

or ⇒ vibrations in wave.

here, focus on short mean-free path ordering.

For $C(f) \gg \partial f/\partial t,\ \underline{v}\cdot\underline{\nabla} f$, etc.

l.o. $\quad C(f) = 0$

$$\Rightarrow f = f_{\text{Maxwellian}}$$

i.e. – collisions drive distribution function to local Maxwellian on timescale short compared all else
– n.b. Maxwellian can be shifted, and have gradients.

1st order:

$$\frac{\partial f^{(0)}}{\partial t} + \underline{v}\cdot\nabla f^{(0)} + \frac{q}{m}\left(\underline{E}+\frac{\underline{v}}{c}\times\underline{B}\right)\cdot\underline{\nabla} f^{(0)} = C(f^{(1)})$$

then integrating:

$$\int d^3v \left[\frac{\partial f^{(0)}}{\partial t} + \nabla\cdot \underline{v} f^{(0)} + \frac{\partial}{\partial \underline{v}}\cdot\left(\frac{q}{m}\left(\underline{E}+\frac{\underline{v}}{c}\times\underline{B}\right) f^{(0)}\right)\right] \quad \text{IBP} \ 0$$

$$= \int d^3v\ C(f^{(1)})$$

Now, $\int d^3v\, C(f_\circ) = 0$ → collisions conserve #!

so, have:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{V}) = 0$$

i.e. continuity equation!

$$\begin{cases} n = \int d^3v\ f \\ \underline{V} \equiv \int d^3v\ \underline{v}f \Big/ n \end{cases}$$ → basic moments.

→ Now first order moment:

$$\int d^3v\, \underline{v} \left(m \frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f m + q \left(\underline{E} + \underline{v} \times \underline{B} \right) \cdot \underline{\nabla}_v f = C(f) \right)$$

$$① = m \frac{\partial}{\partial t} \left(n \underline{V} \right) \qquad \underline{V} = V(\underline{x}, t)$$

$$③ = \int \underline{v} q (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}}$$

$$= \int \frac{\partial}{\partial \underline{v}} \left[f \underline{v} (\underline{E} + \underline{v} \times \underline{B}) \right] d^3 v - \int f \underline{v} \frac{\partial}{\partial \underline{v}} \cdot (\underline{E} + \underline{v} \times \underline{B})$$

$$- \int f (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial \underline{v}}{\partial \underline{v}}$$

So

$$= -qn (\underline{E} + \underline{V} \times \underline{B})$$

$$④ = \int d^3 v\, m C(f)\, \underline{v}$$

$$= \underline{P}_{ij}$$

→ collisional momentum transfer from species i to j

which leaves ②:

$$② = m\int d^3v \ \underline{v}(\underline{v}\cdot\underline{\nabla}) f$$

$$= m\int d^3v \ \underline{\nabla}\cdot(f\underline{v}\,\underline{v})$$

$$= \underline{\nabla}\cdot\left[m\int d^3v\, f\,\underline{v}\,\underline{v}\right] = m\,\underline{\nabla}\cdot(n\,\overline{\underline{v}\underline{v}})$$

clearly useful to separate $\underline{v}$ into mean and fluctuating pieces

$$\underline{v} = \underline{V} + \underline{w}$$

$$\Rightarrow \underline{\nabla}\cdot(n\,\overline{\underline{v}\underline{v}}) = \underline{\nabla}\cdot(n\underline{V}\,\underline{V}) + \underline{\nabla}\cdot(n\,\overline{\underline{w}\underline{w}})$$

$$+ \underline{\nabla}\cdot n\,(\underbrace{\underline{V}\,\overline{\underline{w}} + \overline{\underline{w}}\,\underline{V}}_{0,\ \mathrm{defn.}})$$

$$\underline{\nabla}\cdot(n\underline{V}\,\underline{V}) = \underline{V}\ \underline{\nabla}\cdot(n\underline{V}) + n\,(\underline{V}\cdot\underline{\nabla})\,\underline{V}$$

$$n\,\overline{\underline{w}\,\underline{w}} \equiv \underline{P}$$

pressure tensor!.

so, can write for momentum equation

$$m \frac{\partial}{\partial t}(n\underline{V}) + m\underline{V}\,\underline{\nabla}\cdot(n\underline{V}) + mn(\underline{V}\cdot\underline{\nabla})\underline{V}$$
$$+ \underline{\nabla}\cdot\underline{P} - qn(\underline{E} + \underline{V}\times\underline{B}) = R_{ij}$$

and using continuity:

$$\boxed{mn\left[\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn(\underline{E} + \underline{V}\times\underline{B}) - \underline{\nabla}\cdot\underline{\underline{P}} + \underline{R_{ij}}}$$

Now, for form $\underline{P}$:

$$\underline{\underline{P}} = \int d^3v\; m\, v_i v_j f$$

in short mean-free-path ordering,

$$f \cong f_{Maxwellian}$$

As mean extracted, symmetry $\Rightarrow$

$$\underline{\underline{P}} = \int d^3v\; v_i v_j \delta_{ij}\, f$$

so $\underline{\underline{P}} = \begin{bmatrix} P_1 & & 0 \\ 0 & P_2 & \\ & & P_3 \end{bmatrix}$ pressure tensor diagonal

if isotropic: $P_1 = P_2 = P_3$ (fast $\parallel, \perp$ thermal equilibration)

$\Rightarrow$

$$\underline{\underline{P}} = p \, \underline{\underline{I}}$$

and pressure reduces to scalar, i.e.

$$mn \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \, \underline{V} \right] = qn \left(\underline{E} + \underline{V} \times \underline{B} \right) - \underline{\nabla} p + \underline{R}_{ij}$$

$\Rightarrow$ For second order moment $\rightarrow$ energy (closure $\Leftrightarrow$ energy flux!) $\Rightarrow$ eqn. state

2 species $\Rightarrow$ $p/\rho^\gamma = \text{const.}$

→ Single Fluid (→ MHD)

Can define single fluid variables:

$\rho = n_i M + n_e M \cong n M$ → density

mass velocity:

$\underline{V} = \frac{1}{\rho}\left(n_i M \underline{V}_i + n_e M_e \underline{V}_e\right)$ mean velocity

$\approx \left[\frac{M \underline{V}_i + m_e \underline{V}_e}{M+m}\right] \cong \underline{V}_i$

Current density: relative velocity

$\underline{J} = q\,(n_i \underline{V}_i - n_e \underline{V}_e)$

$\approx n q\,(\underline{V}_i - \underline{V}_e)$, using QN.

Upshot:

- continuity for ions ⇒ single fluid continuity

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{V}) = 0}$$

- adding electron and ion momentum eqns:

$$Mn\left(\frac{\partial \underline{V}_i}{\partial t} + \underline{V}_i \cdot \underline{\nabla}\, \underline{V}_i\right) = qn\left(\underline{E} + \underline{V}_i \times \underline{B}\right) - \underline{\nabla} p_i + \underline{P}_{i,e}$$

$+$

$$m_e n\left(\frac{\partial \underline{V}_e}{\partial t} + \underline{V}_e \cdot \underline{\nabla}\, \underline{V}_e\right) = -qn\left(\underline{E} + \underline{V}_e \times \underline{B}\right) - \underline{\nabla} p_e + \underline{P}_{e,i}$$

$\Rightarrow$

$$n\left(\frac{\partial}{\partial t}(M\underline{V}_i + m\underline{V}_e) + M(\underline{V}_i \cdot \underline{\nabla})\underline{V}_i + m_e(\underline{V}_e \cdot \underline{\nabla})\underline{V}_e\right)$$
$$= qn(\underline{V}_i - \underline{V}_e) \times \underline{B} - \underline{\nabla}(p_i + p_e)$$
$$\underbrace{\cancel{+ \underline{P}_{ei} + \underline{P}_{i,e}}}_{\text{momentum cons.}}$$

as: $m_e \ll M$

$\underline{J}$ defn.

$\rho = \rho_e + \rho_i$

$$\Rightarrow \boxed{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = \underline{J}\times\underline{B} - \underline{\nabla} p + F_{body}}$$

any additional body force.

Momentum balance.

⇒ Now, only re-maining non-trivial MHD equation is Ohm's Law.

→ where the bodies are buried....

Consider, $\left[m_e * (\text{ion momentum eqn}) - M * (\text{electron momentum eqn.}) \right]$

$$\Rightarrow M m_e n \left(\frac{\partial}{\partial t}(\underline{V}_i - \underline{V}_e) + \underline{V}_i\cdot\underline{\nabla}\,V_i - \underline{V}_e\cdot\underline{\nabla}\,\underline{V}_e \right)$$

$$= qn\,(M + m_e)\,\underline{E} + qn\,(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

$$- m\nabla P_i + M\nabla P_e - (M+m)\,\underline{P}_{ei}$$

Now, ① $\underline{P}_{ei}$ = electron-ion momentum transfer

$$= -M n q \eta \underline{J}$$

② $M \gg m_e$

③ neglecting advective derivatives

$\Rightarrow$

$$\frac{M m_e n}{q} \frac{\partial}{\partial t} \left(\frac{\underline{J}}{n} \right) = q \rho \underline{E} - M n q \eta \underline{J} + M \nabla P_e + q n (m \underline{V}_i + M \underline{V}_e) \times \underline{B}$$

and can further simplify:

$$m \underline{V}_i + M \underline{V}_e = M \underline{V}_i + m \underline{V}_e - (M-m)(\underline{V}_i - \underline{V}_e)$$

$$\cong \frac{\rho \underline{V}}{n} - \frac{M}{nq} \underline{J}$$

Finally, re-arrangeing $\Rightarrow$

$$\boxed{\frac{m_e}{nq^2} \frac{\partial \underline{J}}{\partial t} = \left(\underline{E} + \frac{\underline{V} \times \underline{B}}{c} \right) - \eta \underline{J} - \frac{1}{nq} (\underline{J} \times \underline{B}) + \frac{1}{nq} \nabla P_e}$$

Now, have generalized Ohm's Law:

$$\underset{①}{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t}} = \underset{②}{\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{⑤ c}\right)} - \underset{③}{\eta \underline{J}} - \underset{④}{\left(\frac{\underline{J}\times\underline{B}}{nq}\right)} + \frac{\nabla P_e}{nq}$$

② ⇒ ideal MHD Ohm's Law

③ ⇒ collisional resistivity

} resistive MHD

'ring on ④ : Hall Term

⇒ Hall MHD

bring in ⑤ : Electron thermal force / pressure

⇒ diamagnetic / finite electron ω_* MHD

i.e. Boltzmann response : $\underline{E}$ vs $\frac{\nabla P_e}{nq}$

① : Electron inertia term ($\sim m_e$)

⇒ EMHD, electron inertially modified MHD.
($\omega m_e/nq^2 > \eta$)

For low frequency, strong collisionality, etc.

$$\Rightarrow \quad \underline{E} + \underline{v} \times \underline{B} = \eta \underline{J} \quad \Big\} \text{ Resistive MHD.}$$

N.B. :- Ohm's Law is most sensitive part of MHD structure → need care.

- high ω → electron inertia
  tok. μ-inst → thermal force term.
  $\lambda \sim c/\omega_{pi}$ → Hall term.

this is answered most clearly for the case of
incompressible MHD ---

$\underline{\nabla} \cdot \underline{V} = 0$ → defines equation of state
→ sets P_{total} field

($\omega/k \ll C_s, V_{MS}$)
sound → magnetosonic

$$\nabla \cdot \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} = -\frac{\underline{\nabla}}{\rho}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V} = 0$$

so $\rho \to$ constant ρ_0 (can relax to slow variation)

$$\nabla^2 \left[\left(P + \frac{B^2}{8\pi}\right) \Big/ \rho_0 \right] = \underline{\nabla} \cdot \left(\frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla}\, \underline{V} \right)$$

total pressure

a la' Poisson's equation:

$$\frac{P + B^2}{8\pi} = -\int \frac{d^3x'}{4\pi|X - X'|} \left\{ \underline{\nabla}' \cdot \left(\frac{\underline{B} \cdot \underline{\nabla}'\, \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla}'\, \underline{V} \right) \right\}$$

solves for: P_{tot} field → eliminates eqn. state.

$p^* = p_{tot}$

so

$$\boxed{\begin{aligned} &\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\left(\frac{p^*}{\rho_0}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho_0} \\ &\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} - \eta\nabla^2\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} \end{aligned}}$$

with $\underline{\nabla}\cdot\underline{V} = 0$, constitute equations of incompressible MHD.

→ Rather clearly, this system is one of two, dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ Compressible MHD is really a problem in 3 fields, two of which are vectors

i.e.
$$\begin{cases} \underline{V}(\underline{x},t) & \rightarrow \text{fluid velocity} \\ \underline{B}(\underline{x},t) & \rightarrow \text{magnetic field} \\ S(\underline{x},t) & \rightarrow \text{entropy} \Rightarrow \text{energy density} \end{cases}$$

i.e. scalar equation of state provides 3rd field.

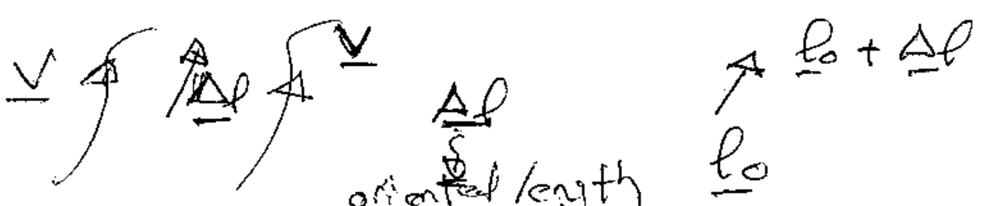
→ Key Question: How closely coupled are $\underline{V}$, $\underline{B}$?

⇒ the key physics element in MHD ....

⇒ $\boxed{\text{Frozen-in Law, Flux Freezing}}$

① Frozen-in Law

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands



i.e. massless rubber strands in flow

How does $\underline{\Delta \ell}$ evolve ?

in dt, $$d(\underline{\Delta \ell}) = \left(\underline{V}(\underline{\ell_0} + \underline{\Delta \ell}) - \underline{V}(\underline{\ell_0})\right) dt$$

$$= \underline{\Delta \ell} \cdot \nabla \underline{V} \quad dt$$

$$\therefore \quad \frac{d(\underline{\Delta \ell})}{dt} = \underline{\Delta \ell} \cdot \nabla \underline{V}$$

i.e. $\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell} \cdot \underline{\underline{S}}$

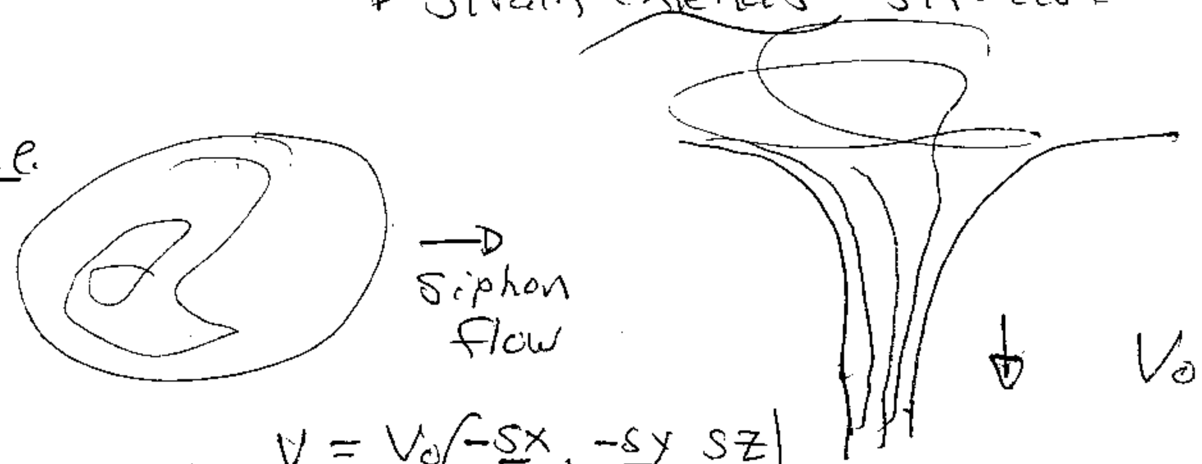
$$\begin{cases} \frac{d}{dt}(\Delta \ell)_i = \underline{\Delta \ell}_j \cdot S_{ij} \\ S_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \end{cases}$$

$\rightarrow$ strain rate tensor

says that $\rightarrow$ $\underline{\Delta \ell}$ strands orient along strain

$\rightarrow$ strain extends strands ....

i.e.



$\underline{V} = V_0\left(-\frac{sx}{2}, -\frac{sy}{2}, sz\right)$

plausible to say that $\underline{\Delta \ell}$ "frozen into" the flow.

Now, if $\eta \rightarrow 0$, ... in MHD

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\; \underline{\nabla} \cdot \underline{V}$$

$$-\underline{\nabla} \cdot \underline{V} = +\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t}\underline{B} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \frac{d\underline{B}}{dt} = \underline{B} \cdot \underline{\nabla}\, \underline{V} + \frac{\underline{B}}{\rho}\frac{d\rho}{dt}$$

$$\frac{1}{\rho}\frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2}\frac{d\rho}{dt} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla} V$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}}$$

→ $\underline{B}/\rho$ obeys same equation as $\underline{\Delta l}$!

→ $\underline{B}/\rho$ is frozen into flow field $\underline{V}(\underline{x}, t)$

Note: → $\underline{B}/\rho$ is <u>not</u> passive → due $\underline{J} \times \underline{B}$ force

→ $\underline{B}$ determines flow, while frozen into it!

→ (essence of coupling problem)

⇒ For $\underline{\nabla} \cdot \underline{V} = 0$, $\underline{B}$ frozen in

⇒ if $\eta \neq 0$, freezing in is broken ----

i.e. $$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \frac{\eta}{\rho} \nabla^2 \underline{B} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla} \underline{V}$$

form of frozen evolution broken

⇒ Observe: → this motivates attention to resistivity in MHD above other dissipations ν, χ, etc..

→ $\eta \Rightarrow \underline{B}$ diffusion $\sim \eta \nabla^2$

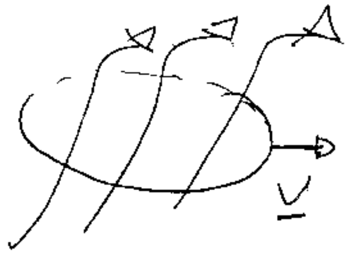
∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

⇒ motivates 'magnetic reconnection' as study of singularity dynamics in MHD.

→ A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

→ Closely Related: Flux Freezing

- consider flux thru surface in flow

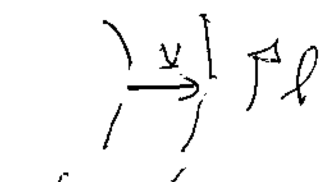


→ i.e. imaginary loop drawn on flow field ...

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

$$\Phi = \int \underline{B} \cdot d\underline{S}$$

$$\frac{d\Phi}{dt} = \underbrace{\int d\underline{S} \cdot \frac{\partial \underline{B}}{\partial t}}_{\text{change in } \underline{B}}^{①} + \underbrace{\int \frac{d\underline{S}}{dt} \cdot \underline{B}}_{\text{motion of loop...}}^{②}$$

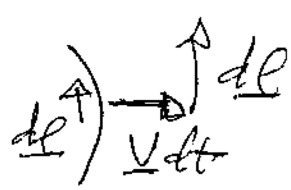


$$① = \int d\underline{S} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$= \oint d\underline{l} \cdot (\underline{V} \times \underline{B})$$

$$d\sigma = \underline{V}\,\Delta t \times d\underline{l}$$

For ②



$\underline{\underline{so}}$ $d\underline{S} = \underline{V}dt \times d\underline{l}$

↳ { change in $\underline{S}$ in dt.

$dt\,(d\Phi_②/dt)$

$$② dt = \int (\underline{V}dt \times d\underline{l}) \cdot \underline{B} = d\Phi$$

$$\left.\frac{d\Phi}{dt}\right)_② = \int (\underline{V} \times d\underline{l}) \cdot \underline{B} = -\int d\underline{l} \cdot (\underline{V} \times \underline{B})$$

so $\frac{d\Phi}{dt} = ① + ②$
$= 0$

so ⇒ magnetic flux invariant ⇔ cancellation

⇒ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

→ no surprise: $\underline{B}$ frozen in ⇒ $\underline{\Phi}$ frozen in

⇒ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \qquad \omega = \underline{\nabla} \times \underline{v}$$

In inviscid hydro, $(\nu \to 0)$ circulation Γ_c is conserved.

Exercise: Prove this!
Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = \text{const.}$, $\underline{g} = 0$

Extra Credit: ① Discuss the extension to the case where $\rho \neq \text{const.}$
② What is 'frozen in' for Vlasov plasma?

→ What Does "Freezing" Mean?

⇒ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

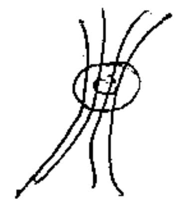
Application :

→ proto-stellar cloud

Star formation

$\underline{B}$-field

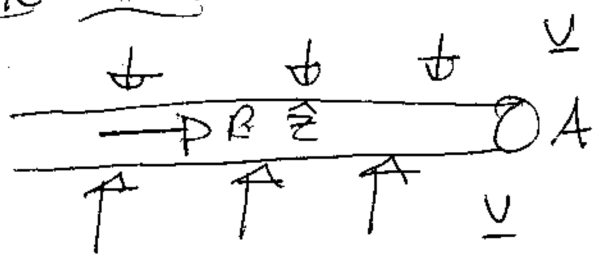
→



(?)

Simple Cases → How does $\underline{B}$ change in a flow?

(i)



radial compression

{ Cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\,\hat{z}$

2 ways:

$$\frac{d}{dt}\left(B\hat{z}/\rho\right) = \frac{\underline{B}\hat{z}}{\rho}\cdot\underline{\nabla}\,\underline{V}$$

$$= 0$$

$\underline{V} = v\,\hat{r}$

$\Rightarrow \underline{V} \perp \underline{B}$

so $\underline{B}/\rho = \text{const}$

Now: $\rho A L = \text{const}$ so $B \sim A^{-1}$

$\rho \sim A^{-1}$

(L const.)

or

Flux Frozen: $BA = \Phi = \text{const.}$

$\rho AL = \text{const} = M$

L const.

$BA \sim \Phi_0$, $B \sim A^{-1}$

$\rho A \sim M_0$, $\rho \sim A^{-1}$

so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim \text{const}!$

$V = V(z)\hat{z}$ — compressible!

ii.) $\longrightarrow B\hat{z}$ i.e. stretch, 1D

here $\dfrac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial B}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

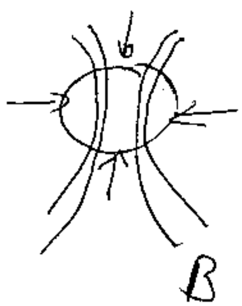
$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

$$= 0 \quad !$$

For ρ, $\frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V} = -\rho \frac{\partial V_z}{\partial z}$

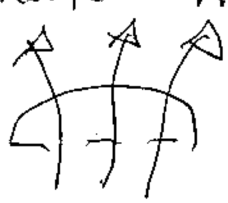
here $\underline{B}$ invariant, ρ changes

<u>i.e.</u> $B \sim \rho^{(0)}$

iii.)  collapsing sphere: $\underline{V} = V \hat{r}$

 (<u>i.e.</u> $\Phi = 0$ total sphere)

consider hemispherical surface (i.e. mushroom cap)



$\Phi \sim B R^2 \sim$ const

$M \sim \rho R^3 \sim$ const.

$\Rightarrow$ $B \sim r^{-2}$ $\Rightarrow$ $B/\rho^{2/3} \sim$ const.
$\rho \sim r^{-3}$

why the scaling ? $\Leftrightarrow$ why of interest ?

→ {"implosion", gravitational collapse} problems sensitive to equation of state of material collapsing

collapse threaded by magnetic field

$\underline{\text{If}}$: $P \to P_{tot} = P + \frac{B^2}{8\pi}$

$$P = P_0 (\rho/\rho_0)^{\gamma}$$

then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution?

⇒ Proceed via flux-freezing!

$$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$$

$\underline{\text{so}}$ P_{B^2} has "$\gamma_{eff} = 4/3$". This resembles equation of state for degenerate gas (see Handouts I).

⇒ More on this in discussion of flux freezing and virial theorems ....

→ Pragmatic Question: Is flux 'frozen' during star formation? ⟷ Does resistivity matter?

$$\eta \sim \frac{4\times10^{6}}{T_{eV}^{3/2}}\ \mathrm{cm^2/sec.} \qquad \text{(Spitzer)}$$

start $\rightarrow$ collapse $\rightarrow$ protostar

$n \sim 1\ \text{atom}/\mathrm{cm}^3 \longrightarrow \rho \sim 1\,\mathrm{g/cm^3}$, $n \sim 10^{24}\ \text{atm}/\mathrm{cm}^3$ (related N_A)

but

$B/\rho^{2/3} \sim$ const

$\Rightarrow \quad B/B_0 \sim \left(10^{24}\right)^{2/3} \sim 10^{16}$! huge amplification

so $B_0 \sim 10^{-6}\ G$, characteristic of ISM

$\Rightarrow B \sim 10^{10}\ G$ in protostar

$\therefore \quad P_{B^2} \sim 10^{19}\ \mathrm{erg/cm^3} \qquad (P_{B^2} \sim B^2/8\pi)$

but P_{Th} for normal star $\sim 10^{14}\ \mathrm{erg/cm^3}$

$P_{B^2} \gg P_{Th}$?? $\Rightarrow$ clearly flux-freezing is bad assumption.

→ In terms of time scales:

$$\underset{①}{\frac{\partial \underline{B}}{\partial t}} = \underset{②}{\underline{\nabla} \times (\underline{v} \times \underline{B})} + \underset{③}{\eta \nabla^2 \underline{B}}$$

$$\frac{1}{T_{collapse}} \sim \frac{1}{T_{dynamic}} + \underset{\substack{\S \\ 1/T_{diffn.}}}{\frac{\eta}{L^2}}$$

3 scales, 2 balance

i.e. ① & ② ③ negligible

① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn} \rightarrow$ {flux frozen, OK

$T_{collapse} \gg T_{diffn} \rightarrow$ {must consider diffusion, freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

but ISM has large neutral component.

Plasma-neutral drag sets dissipation

→ Ambipolar diffusion.